Mathematics Creative Problem Solving Competition

Semi-final (Secondary)

Marking Scheme

1a) When higher order terms are added, the polynomial gets closer to the formula at a given range.	[2]
1b) Yes!	[1]
1c) a_0 = formula at x=0.	[2]

2a) 1. a_0=f(0) 2. Adjust a_i until minimum "error" reached. (i=1 to n) 3. Repeat 2 until i = n	[5]
2b) ➢ a_0 is the y intercept. ➢ a_1 is the slope of tangent at x=0. ➢ a_2 is the curvature of the curve at x=0. Positive values mean concave upward. Negative value mean concave downward.	[1] [1] [2]
2c) ➢ Visual observation ➢ Minimum area ➢ Minimum difference at predefined points ➢ Minimum difference square at predefined points OR any other reasonable suggestions.	[2]
2d) Increase number of higher order terms. (i.e. increase the degree of the polynomial)	[2]
2e) i. The area around x = 0 has smaller error. Error increases as it approaches *x* = 1.5.	[2]
2e) ii. 1. <u>Shift the function to left by 0.5. (X=x-0.5)</u> 2. a_0=f(X=0) 3. Adjust a_i until minimum "error" reached *around X=0.* (i=1 to n) 4. Repeat 3 until i = n 5. <u>After finding all coefficients a_i, the new coefficient of polynomial can be obtained by shifting the function right by 0.5 (x=X+0.5) and expanding the polynomial. i.e. $y = a_0 + a_1(x+0.5) + a_2(x+0.5)^2 + a_3(x+0.5)^3 + a_4(x+0.5)^4 + a_5(x+0.5)^5$</u> [Logical and systematic presentation]	[2] [2] [1]

3) 1. $a_o = f(0) = -1$ 2. From observations, a_1 is equal to the slope of the tangent. The tangent is approximated by = $(f(\delta x) - f(-\delta x))/\ 2\delta x$, where δx is a small number. i.e. $a_1 \approx (f(\delta x) - f(-\delta x))/\ 2\delta x$ 3. $f(x) \approx a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, $f(x) - (a_0 + a_1 x) \approx a_2 x^2 + \cdots + a_n x^n$ In order to find a_2, set x to be a small number and neglect those order 3 or higher terms.	
Logical approach	[3]
Systematic presentation	[1]
Illustration of using this method to solve (3): $$a_o = f(0) = -1$$ For demonstration, set $\delta x = 0.01$. $$a_1 \approx \frac{f(+\delta x) - f(-\delta x)}{2\delta x} = 1.99980002$$ $$f(x) - (a_0 + a_1 x) \approx a_2 x^2 + \cdots + a_n x^n$$ $$f(x) - (-1 + 1.99980002x) \approx a_2 x^2 + \cdots + a_n x^n$$ set x=0.01 and neglect those order 3 and higher terms. -0.97990201 – (-0.98) $\approx a_2 0.01^2$ $$a_2 \approx 0.99990001$$ Similarly, if $\delta x = 0.001$, $a_1 \approx 1.999998$, $a_2 \approx 0.999999$ $\delta x = 0.0001$, $a_1 \approx 1.99999998$, $a_2 \approx 0.999999983$ ** It can be shown by Taylor expansion that: $$\frac{2x - 1}{x^2 + 1} = -1 + 2x + x^2 - 2x^3 - x^4 + 2x^5 + x^6 + \cdots$$ NOTE! BEWARE OF TRUCATION ERROR! $\delta x = 0.000000001$, $a_1 \approx 1.999999999$, $a_2 \approx 111.0223025$	
Successful demonstration and explanation.	[2]

(4) Pros: Simple & yet give better mathematical intuition Accuracy can be improved by adding higher order terms. Cons: The approximation is good for only small displacement from the expansion point. Otherwise, new expansions are needed. No good for periodic functions.	[2]

Digression:

Taylor expansion

$$f(x) \approx f(x_0) + \frac{f^{(1)}(x_0)}{1!}(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

$$y_1 = \frac{1}{2 - x} \approx \frac{1}{2} + \frac{1}{4}x + \frac{1}{8}x^2 + \frac{1}{16}x^3 + \frac{1}{32}x^4 + \frac{1}{64}x^5$$

$$= 0.5 + 0.25x + 0.125x^2 + 0.0625x^3 + 0.03125x^4 + 0.015625x^5$$

$$y_2 = \frac{10x^3 + 5x^2 + 1}{x^2 + 5x + 10} \approx \frac{1}{10} - \frac{1}{20}x + \frac{103}{200}x^2 + \frac{299}{400}x^3 - \frac{1701}{4000}x^4 + \frac{1103}{8000}x^5$$

$$= 0.1 - 0.05x + 0.515x^2 + 0.7475x^3 - 0.42525x^4 + 0.13785x^5$$

$$y_3 = \frac{1 - x + \frac{x^2}{2} - \frac{x^3}{6}}{x^2 - 4} \approx -\frac{1}{4} + \frac{1}{4}x - \frac{3}{16}x^2 + \frac{5}{48}x^3 - \frac{3}{64}x^4 + \frac{5}{192}x^5$$

$$= -0.25 + 0.25x - 0.1875x^2 + 0.104166x^3 - 0.046875x^4$$
$$+ 0.0260416x^5$$

Extra Questions:

Why should we start with x_0 instead of x_5? What is the reasonable range of x-x_0?

Are there any restrictions on setting δx? How to find them out?

Give one example that this kind of expansion doesn't work.

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